

Variable-width confidence intervals in Gaussian regression and penalized maximum likelihood estimators

Davide Farchione and Paul Kabaila*

Department of Mathematics and Statistics, La Trobe University, Australia

* Author to whom correspondence should be addressed. Department of Mathematics and Statistics, La Trobe University, Victoria 3086, Australia. Tel.: +61 3 9479 2594, Fax: +61 3 9479 2466, E-mail: P.Kabaila@latrobe.edu.au

ABSTRACT

Hard thresholding, LASSO , adaptive LASSO and SCAD point estimators have been suggested for use in the linear regression context when most of the components of the regression parameter vector are believed to be zero, a sparsity type of assumption. Pötscher and Schneider, 2010, *Electronic Journal of Statistics*, have considered the properties of fixed-width confidence intervals that include one of these point estimators (for all possible data values). They consider a normal linear regression model with orthogonal regressors and show that these confidence intervals are longer than the standard confidence interval (based on the maximum likelihood estimator) when the tuning parameter for these point estimators is chosen to lead to either conservative or consistent model selection. We extend this analysis to the case of *variable-width* confidence intervals that include one of these point estimators (for all possible data values). In consonance with these findings of Pötscher and Schneider, we find that these confidence intervals perform poorly by comparison with the standard confidence interval, when the tuning parameter for these point estimators is chosen to lead to consistent model selection. However, when the tuning parameter for these point estimators is chosen to lead to conservative model selection, our conclusions differ from those of Pötscher and Schneider. We consider the variable-width confidence intervals of Farchione and Kabaila, 2008, *Statistics & Probability Letters*, which have advantages over the standard confidence interval in the context that there is a belief in a sparsity type of assumption. These variable-width confidence intervals are shown to include the hard thresholding, LASSO, adaptive LASSO and SCAD estimators (for all possible data values) provided that the tuning parameters for these estimators are chosen to belong to an appropriate interval.

1 Introduction

Hard-thresholding, LASSO (Tibshirani [7]), adaptive LASSO (Zou [8]) and SCAD (Fan and Li [1]) point estimators have been suggested for use in the linear regression context when most of the components of the regression parameter vector are believed to be zero, a sparsity type of assumption. Pötscher and Schneider [5] ask to what extent these point estimators can be used as the basis for confidence intervals for these components. They consider the properties of fixed-width confidence intervals that are constrained to include one of these point estimators (for all possible data values). They do this in the context of a normal linear regression model with orthogonal regressors for both the case that (a) the error variance is assumed known and (b) the error variance is estimated by the usual unbiased estimator obtained by fitting the full model to the data. Pötscher and Schneider [5] show that these confidence intervals are longer than the standard confidence interval based on the maximum likelihood estimator, when the tuning parameter for these point estimators is chosen to lead to either conservative or consistent model selection. By consistent model selection, we mean that the selected model is the true model with probability approaching 1 as $n \rightarrow \infty$, where n denotes the dimension of the response vector. By conservative model selection, we mean a model selection that (a) is not consistent and (b) is such that the selected model includes the true model with probability approaching 1 as $n \rightarrow \infty$.

To what extent are these findings due to the requirement that these confidence intervals have fixed widths? A variable-width confidence interval based on a given point estimator has the property that this confidence interval includes this point estimator, for all possible data values. We first consider the case that the tuning parameter for these point estimators is chosen to lead to consistent model selection. In Section 3, we present a new result that shows that variable-width confidence intervals that include one of these point estimators (for all possible data values)

must perform poorly by comparison with the standard confidence interval. In this case, our conclusions are similar to those in [5]. This is perhaps not surprising, given the results of Kabaila [3] and Pötscher [4].

Next, we consider the case that the tuning parameter for these point estimators is chosen to lead to conservative model selection. Pötscher and Schneider [5] find that fixed-width confidence intervals that are constrained to include one of these point estimators (for all possible data values) are longer than the standard confidence interval. This may be interpreted as a negative finding for these point estimators. Yet, these point estimators have some very attractive features. Figure 9 of [7] shows contours of constant value of $|\beta_1|^q + |\beta_2|^q$ for $q = 4, 2, 1, 0.5$ and 0.1 . As Tibshirani [7] states, “The lasso corresponds to $q = 1$.” and “The value $q = 1$ has the advantage of being closer to subset selection than is ridge regression ($q = 2$) and is also the smallest value of q giving a convex region.”. The LASSO estimator has the attractive feature that it is a continuous function of the data. Like the LASSO, the adaptive LASSO and the SCAD estimators use a thresholding rule that sets estimated coefficients with small magnitudes to zero. The adaptive LASSO and the SCAD estimators also have the attractive features that (a) they are continuous functions of the data and (b) they are nearly unbiased when the true unknown parameter has large magnitude ([1], [8]). How do we resolve the apparent conflict between the findings of [5] and the existence of these very attractive features? We show that this finding can be explained (at least in part) by the requirement in [5] that the confidence intervals have fixed widths.

Following [5], we consider a normal linear regression model with orthogonal regressors for both the case that (a) the error variance is assumed known and (b) the error variance is estimated by the usual unbiased estimator obtained by fitting the full model to the data. It is plausible that the case that the error variance is known amounts essentially to the assumption that the error variance is estimated with great

accuracy. In Appendix B, we provide a precise motivation for considering the known error variance case. In Section 4, we consider the variable-width confidence intervals of Farchione and Kabaila [2], in the known error variance case. These confidence intervals are shown to have advantages over the standard confidence interval when there is a belief in a sparsity type of assumption. These variable-width confidence intervals are shown to include the hard-thresholding, LASSO, adaptive LASSO and SCAD estimators (for all possible data values) provided that the tuning parameters for these estimators are chosen to belong to an appropriate interval. In Section 5, we consider the extension of these results to the case that the error variance is estimated by the usual unbiased estimator obtained by fitting the full model to the data.

2 The model and the point estimators considered

We consider a normal linear regression model with orthogonal regressors. As pointed out in [5], without loss of generality we may suppose that the data Y_1, \dots, Y_n are independent and identically $N(\theta, \sigma^2)$ distributed, where $\theta \in \mathbb{R}$ and $\sigma > 0$. We use lower case to denote the observed value of a random variable. We also use a similar notation to that used in [5] for the hard thresholding, LASSO and adaptive LASSO estimators. Namely, the hard thresholding estimator $\tilde{\Theta}_H$ is given by

$$\tilde{\Theta}_H = \bar{Y} \mathbf{1}(|\bar{Y}| > \hat{\Sigma}\eta_n) = \begin{cases} 0 & \text{if } |\bar{Y}| \leq \hat{\Sigma}\eta_n \\ \bar{Y} & \text{if } |\bar{Y}| > \hat{\Sigma}\eta_n \end{cases}$$

where the tuning parameter η_n is a positive real number, $\bar{Y} = n^{-1} \sum_{i=1}^n Y_i$ and $\hat{\Sigma}^2 = (n-1)^{-1} \sum_{i=1}^n (Y_i - \bar{Y})^2$. The LASSO estimator $\tilde{\Theta}_S$ is given by

$$\tilde{\Theta}_S = \text{sign}(\bar{Y}) (|\bar{Y}| > \hat{\Sigma}\eta_n)_+ = \begin{cases} -\max\{|\bar{Y}| - \hat{\Sigma}\eta_n, 0\} & \text{if } \bar{Y} < 0 \\ 0 & \text{if } \bar{Y} = 0 \\ \max\{|\bar{Y}| - \hat{\Sigma}\eta_n, 0\} & \text{if } \bar{Y} > 0 \end{cases}$$

where $\text{sign}(x)$ is equal to -1 for $x < 0$, 0 for $x = 0$ and 1 for $x > 0$ and $x_+ = \max\{x, 0\}$. The adaptive LASSO estimator $\tilde{\Theta}_A$ is given by

$$\tilde{\Theta}_A = \bar{Y} (1 - \hat{\Sigma}^2 \eta_n^2 / \bar{Y}^2)_+ = \begin{cases} 0 & \text{if } |\bar{Y}| \leq \hat{\Sigma} \eta_n \\ \bar{Y} - \frac{\hat{\Sigma}^2 \eta_n^2}{\bar{Y}} & \text{if } |\bar{Y}| > \hat{\Sigma} \eta_n \end{cases}$$

We also consider the following SCAD estimator $\tilde{\Theta}_C$

$$\tilde{\Theta}_C = \begin{cases} \text{sign}(\bar{Y}) (|\bar{Y}| - \hat{\Sigma} \eta_n)_+ & \text{if } |\bar{Y}| \leq 2\hat{\Sigma} \eta_n \\ ((a-1)\bar{Y} - \text{sign}(\bar{Y})a\hat{\Sigma} \eta_n)/(a-2) & \text{if } 2\hat{\Sigma} \eta_n < |\bar{Y}| \leq a\hat{\Sigma} \eta_n \\ \bar{Y} & \text{if } |\bar{Y}| > a\hat{\Sigma} \eta_n \end{cases}$$

where $a = 3.7$ (see p.1351 of [1] for a motivation for this choice of a).

3 Variable-width confidence intervals based on the point estimators when the tuning parameter is chosen for consistent model selection

In this section, we suppose that $\eta_n \rightarrow 0$ and $\sqrt{n} \eta_n \rightarrow \infty$, as $n \rightarrow \infty$. In other words, we suppose that the tuning parameter η_n is chosen so as to lead to consistent model selection. In this case, for example, the probability that $\tilde{\Theta}_H$ is equal to 0 approaches 1 for $\theta = 0$, whilst $\tilde{\Theta}_H$ converges in probability to θ for $\theta \neq 0$ (as $n \rightarrow \infty$). For clarity, in this section we will use the subscript n to make explicit a dependence on n . Let $\tilde{\theta}_n(\bar{y}_n, \hat{\sigma}_n)$ denote a point estimate of θ that satisfies the condition that if $|\bar{y}_n| \leq \hat{\sigma}_n \eta_n$ then $\tilde{\theta}_n(\bar{y}_n, \hat{\sigma}_n) = 0$. The estimates $\tilde{\theta}_H$, $\tilde{\theta}_S$ and $\tilde{\theta}_A$ satisfy this condition. With a small change of notation, the estimate $\tilde{\theta}_C$ also satisfies this condition. The standard $1 - \alpha$ confidence interval for θ is

$$J_n = \left[\bar{Y}_n - t(n-1)\hat{\Sigma}_n/\sqrt{n}, \bar{Y}_n + t(n-1)\hat{\Sigma}_n/\sqrt{n} \right]$$

where the quantile $t(m)$ is defined by the requirement that $P(-t(m) \leq T \leq t(m)) = 1 - \alpha$ for $T \sim t_m$.

A variable-width confidence interval based on the point estimate $\tilde{\theta}_n(\bar{y}_n, \hat{\sigma}_n)$ has the property that this confidence interval includes this point estimate, for all possible

data values. Consider the confidence interval

$$D_n(\bar{Y}_n, \hat{\Sigma}_n) = [\ell_n(\bar{Y}_n, \hat{\Sigma}_n), u_n(\bar{Y}_n, \hat{\Sigma}_n)]$$

for θ , that is required to satisfy the following conditions for all n :

- (a) $\tilde{\theta}_n(\bar{y}_n, \hat{\sigma}_n) \in D_n(\bar{y}_n, \hat{\sigma}_n)$ for all $(\bar{y}_n, \hat{\sigma}_n) \in \mathbb{R} \times (0, \infty)$. In other words, the confidence interval D_n contains the estimate $\tilde{\theta}_n$, for all possible data values.
- (b) $P_{\theta, \sigma}(\theta \in D_n(\bar{Y}_n, \hat{\Sigma}_n)) \geq 1 - \alpha$ for all $(\theta, \sigma) \in \mathbb{R} \times (0, \infty)$. In other words, D_n is a $1 - \alpha$ confidence interval for θ .

The following result shows that this confidence interval performs very poorly by comparison with J_n , the standard $1 - \alpha$ confidence interval for θ .

Theorem 1. *Let $\theta_n = \sigma \eta_n/2$. For each $\sigma \in (0, \infty)$,*

$$\frac{E_{\theta_n, \sigma}(\text{length of } D_n(\bar{Y}_n, \hat{\Sigma}_n))}{E_{\theta_n, \sigma}(\text{length of standard } 1 - \alpha \text{ confidence interval } J_n)} \rightarrow \infty$$

as $n \rightarrow \infty$.

The proof of this theorem is presented in Appendix A.

4 Variable-width confidence intervals of Farchione and Kabaila when the error variance is known

Consider the “known error variance case”. The motivation for considering this case is given in Appendix B. Suppose that σ^2 is known. Consider the $1 - \alpha$ confidence interval for θ , put forward by Farchione and Kabaila [2], that has the form

$$C = \left[-\frac{\sigma}{\sqrt{n}} b\left(-\frac{\bar{Y}}{\sigma/\sqrt{n}}\right), \frac{\sigma}{\sqrt{n}} b\left(\frac{\bar{Y}}{\sigma/\sqrt{n}}\right) \right] \quad (1)$$

where the function b satisfies $b(x) \geq -b(-x)$ for all $x \in \mathbb{R}$. This constraint is required to ensure that the upper endpoint of this confidence interval is never less

than the lower endpoint. This particular form of confidence interval is motivated by the invariance arguments presented in Section 4 of [2]. The standard $1 - \alpha$ confidence interval for θ is $I = [\bar{Y} - z\sigma/\sqrt{n}, \bar{Y} + z\sigma/\sqrt{n}]$, where the quantile z is defined by the requirement that $P(-z \leq Z \leq z) = 1 - \alpha$ for $Z \sim N(0, 1)$. Note that this confidence interval can be expressed in the form C .

The coverage probability and expected length properties of the confidence interval C are conveniently examined by applying the same change of scale (by multiplying by \sqrt{n}/σ) to the parameter θ , the estimator \bar{Y} , the confidence interval C and the standard confidence interval I . Define $\psi = (\sqrt{n}/\sigma)\theta$, $X = (\sqrt{n}/\sigma)\bar{Y}$,

$$C^* = \frac{\sqrt{n}}{\sigma}C = [-b(-X), b(X)], \quad (2)$$

and $I^* = (\sqrt{n}/\sigma)I = [X - z, X + z]$. Note that $X \sim N(\psi, 1)$. We consider C^* to be a confidence interval for ψ , based on X . The standard $1 - \alpha$ confidence interval for ψ (based on X) is I^* . Note that $P_{\theta, \sigma}(\theta \in C) = P_{\psi}(\psi \in C^*)$ and

$$\frac{E_{\theta, \sigma}(\text{length of } C)}{\text{length of } I} = \frac{E_{\psi}(\text{length of } C^*)}{\text{length of } I^*},$$

for $\psi = (\sqrt{n}/\sigma)\theta$.

Following [2], we assess C^* , for parameter value ψ , using the relative efficiency

$$e(\psi) = \left(\frac{E_{\psi}(\text{length of } C^*)}{\text{length of } I^*} \right)^2 = \left(\frac{E_{\psi}(\text{length of } C^*)}{2z} \right)^2.$$

This is a measure of the efficiency of the standard $1 - \alpha$ confidence interval I^* by comparison with the efficiency of the $1 - \alpha$ confidence interval C^* . The relative efficiency $e(\psi)$ is the ratio (sample size used for C^*)/(sample size used for I^*) such that $E_{\psi}(\text{length of } C^*) = \text{length of } I^*$ (cf p.555 of [6]). Farchione and Kabaila [2] use the methodology of Pratt [6], with a new weight function determined by a parameter w , to find a confidence interval C^* such that $e(0)$ is minimized, while ensuring that $\max_{\psi} e(\psi)$ is not too large. In other words, if ψ happens to be 0 then C^* performs better than the standard $1 - \alpha$ confidence interval I^* . On the other hand, if $\psi \neq 0$

then the worst possible performance of C^* is $\max_{\psi} e(\psi)$, which is not too large. In addition, this confidence interval has endpoints that approach the endpoints of the standard $1 - \alpha$ confidence interval I^* as $|x| \rightarrow \infty$. This implies that $e(\psi) \rightarrow 1$ as $|\psi| \rightarrow \infty$. We have chosen $w = 0.1$ and $1 - \alpha = 0.95$. The coverage probability $P_{\psi}(\psi \in C^*)$ is 0.95 for all ψ . The relative efficiency $e(\psi)$ of C^* for this case is shown in Figure 1. For comparison, the 0.95 confidence interval described on p.555 of [6] has relative efficiency 0.72 at $\psi = 0$. This, however, comes at the very high cost of the relative efficiency diverging to ∞ as $|\psi| \rightarrow \infty$.

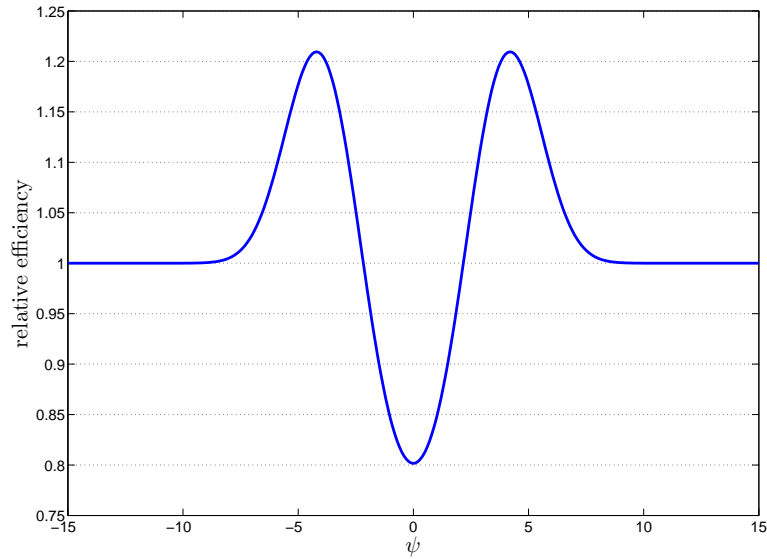


Figure 1: *Plot of the efficiency of the standard 95% confidence interval by comparison with the Farchione and Kabaila 95% confidence interval (for $w = 0.1$) as a function of ψ .*

We now consider the properties of the confidence interval C^* in the context that most of the components of the regression parameter vector are believed to be zero, a sparsity type of assumption. Firstly, suppose that a large majority of the components of the regression parameter vector are zero. In this case, C^* compares very favourably with the standard $1 - \alpha$ confidence interval. If $\psi = 0$, corresponding to one of the large majority of the components of the regression parameter vector that are zero, then $e(\psi)$ is approximately 0.8. On the other hand, if $\psi \neq 0$, correspond-

ing to one of the small minority of components of the regression parameter vector that are non-zero, then the maximum possible value of $e(\psi)$ is approximately 1.2. Secondly, in the “best of all possible worlds” scenario that a large majority of the components of the regression parameter vector are zero and the remaining components have large magnitudes, C^* may be said to effectively dominate the standard $1 - \alpha$ confidence interval. If $\psi = 0$, corresponding to one of the large majority of the components of the regression parameter vector that is zero, then $e(\psi)$ is approximately equal to 0.8. On the other hand, if $|\psi|$ is large, corresponding to one of the small minority of the components of the regression parameter vector that has large magnitude, then $e(\psi)$ is approximately equal to 1. We conclude that C^* has advantages over the standard $1 - \alpha$ confidence interval I^* when a sparsity type of assumption holds.

5 Variable-width confidence intervals based on the point estimators when the tuning parameter is chosen for conservative model selection and the error variance is known

In this section, we suppose that $\eta_n \rightarrow 0$. We also suppose that there exists a positive integer N and a_ℓ and a_u (satisfying $0 < a_\ell < a_u < \infty$), such that $\sqrt{n}\eta_n \in [a_\ell, a_u]$ for all $n > N$. This includes the particular case that $\sqrt{n}\eta_n \rightarrow a$ ($0 < a < \infty$), as $n \rightarrow \infty$. In other words, we suppose that the tuning parameter η_n is chosen so as to lead to conservative model selection. We consider the “known error variance case”. The motivation for considering this case is given in Appendix B.

Suppose that σ^2 is known. We consider the conditions under which the point estimate

$$\hat{\theta}_H = \begin{cases} 0 & \text{if } |\bar{y}| \leq \sigma \eta_n \\ \bar{y} & \text{if } |\bar{y}| > \sigma \eta_n \end{cases}$$

of θ belongs in the confidence interval C (defined by (1)) for all $(\bar{y}, \sigma) \in \mathbb{R} \times (0, \infty)$.

Define $\tau_n = \sqrt{n}\eta_n$. As in Section 4, multiply the estimate $\hat{\theta}_H$ and the confidence interval C by \sqrt{n}/σ , to obtain

$$\hat{\psi}_H = \frac{\sqrt{n}}{\sigma}\hat{\theta}_H = \begin{cases} 0 & \text{if } |x| \leq \tau_n \\ x & \text{if } |x| > \tau_n \end{cases}$$

and $C^* = (\sqrt{n}/\sigma)C$ (see (2)). Obviously, $\hat{\theta}_H \in C$ for all $(\bar{y}, \sigma) \in \mathbb{R} \times (0, \infty)$ is equivalent to $\hat{\psi}_H \in C^*$ for all $x \in \mathbb{R}$. There exists a positive number c_H such that, for every $\tau_n \in (0, c_H]$, the following is true: $\hat{\psi}_H \in C^*$ for all $x \in \mathbb{R}$. Similar statements hold for the other point estimates $\hat{\theta}_S$, $\hat{\theta}_A$ and $\hat{\theta}_C$ (the corresponding estimators are defined towards the end of Appendix B).

Define $\hat{\psi}_S = (\sqrt{n}/\sigma)\hat{\theta}_S$, $\hat{\psi}_A = (\sqrt{n}/\sigma)\hat{\theta}_A$ and $\hat{\psi}_C = (\sqrt{n}/\sigma)\hat{\theta}_C$. We have computed the maximum values of τ_n such that $\hat{\psi}_H$, $\hat{\psi}_S$, $\hat{\psi}_A$ and $\hat{\psi}_C$ are in the interval C^* (for all x). In each case this maximum value was found to be 1.96. Figures 2 and 3 show the values of the estimator as a function of x for this maximum value, together with the endpoints of the confidence interval C^* as functions of x .

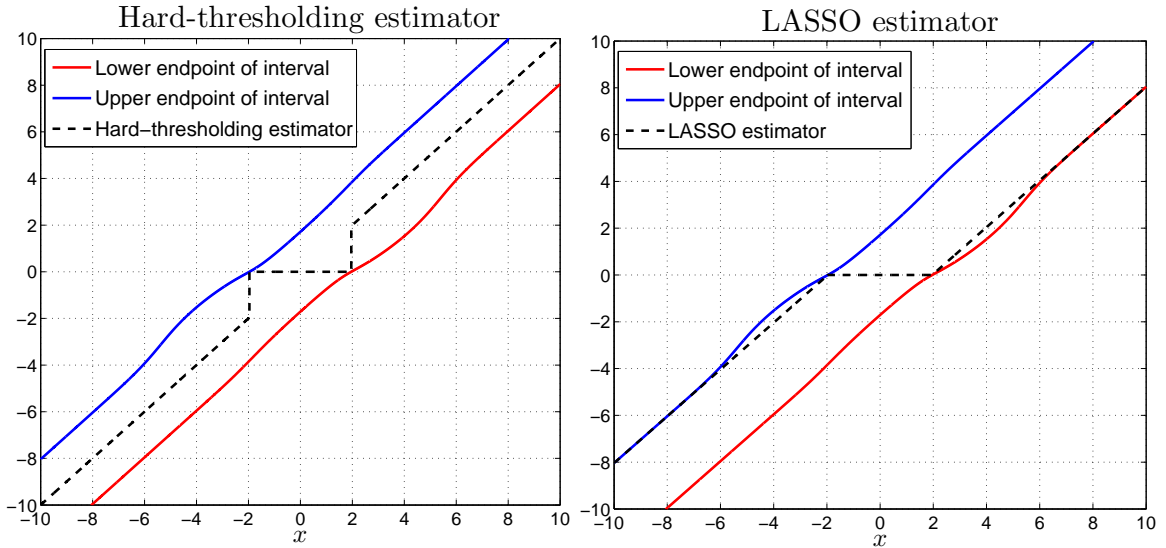


Figure 2: The left and right panels show the hard-thresholding estimate $\hat{\psi}_H$ and the LASSO estimate $\hat{\psi}_S$, respectively, as functions of x (for $\tau_n = 1.96$). Also shown, in both panels, is the Farchione and Kabaila 95% confidence interval C^* as a function of x (for $w = 0.1$).

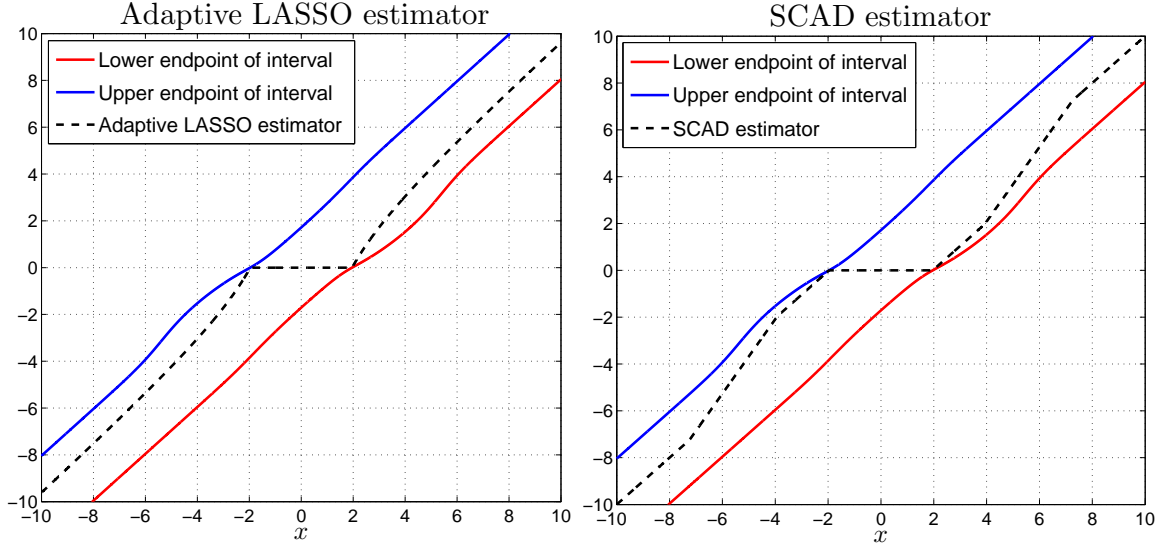


Figure 3: The left and right panels show the Adaptive LASSO estimate $\hat{\psi}_A$ (for $\tau_n = 1.96$) and the SCAD estimate $\hat{\psi}_S$ (for $\tau_n = 1.96$, $a = 3.7$), respectively, as functions of x . Also shown, in both panels, is the Farchione and Kabaila 95% confidence interval C^* as a function of x (for $w = 0.1$).

6 Variable-width confidence intervals of Farchione and Kabaila and the point estimators when the tuning parameter is chosen for conservative model selection and the error variance is unknown

Suppose that the error variance σ^2 is unknown. Consider the $1 - \alpha$ confidence interval for θ , put forward in Section 5 of [2], that has the form

$$D = \left[-\frac{\hat{\Sigma}}{\sqrt{n}} b\left(-\frac{\bar{Y}}{\hat{\Sigma}/\sqrt{n}}\right), \frac{\hat{\Sigma}}{\sqrt{n}} b\left(\frac{\bar{Y}}{\hat{\Sigma}/\sqrt{n}}\right) \right]$$

where the function b satisfies $b(x) \geq -b(-x)$ for all $x \in \mathbb{R}$. This constraint is required to ensure that the upper endpoint of this confidence interval is never less than the lower endpoint. This particular form of confidence interval can be motivated by invariance arguments similar to those presented in Section 4 of [2]. The standard $1 - \alpha$ confidence interval for θ is

$$J = [\bar{Y} - t(n-1)\hat{\Sigma}/\sqrt{n}, \bar{Y} + t(n-1)\hat{\Sigma}/\sqrt{n}]$$

where the quantile $t(m)$ is defined by the requirement that $P(-t(m) \leq T \leq t(m)) = 1 - \alpha$ for $T \sim t_m$. Note that this confidence interval can be expressed in the form D .

Define $R = \hat{\Sigma}/\sigma$. The coverage probability and expected length properties of the confidence interval D are conveniently examined by applying the same change of scale (by multiplying by \sqrt{n}/σ) to the parameter θ , the estimator \bar{Y} , the confidence interval D and the standard confidence interval J . Define $\psi = (\sqrt{n}/\sigma)\theta$, $X = (\sqrt{n}/\sigma)\bar{Y}$,

$$D^* = \frac{\sqrt{n}}{\sigma}D = [-Rb(-X/R), Rb(X/R)].$$

and $J^* = (\sqrt{n}/\sigma)J = [X - t(n-1)R, X + t(n-1)R]$. Note that X and R are independent random variables and that $X \sim N(\psi, 1)$. As noted in Appendix B, the coverage probability and expected length properties of D are conveniently evaluated using the fact that

$$P_{\theta,\sigma}(\theta \in D) = P_{\psi}(\psi \in D^*),$$

and

$$\frac{E_{\theta,\sigma}(\text{length of } D)}{E_{\theta,\sigma}(\text{length of } J)} = \frac{E_{\psi}(\text{length of } D^*)}{E_{\theta,\sigma}(\text{length of } J^*)}$$

for $\psi = (\sqrt{n}/\sigma)\theta$.

Following [2], we assess D^* , for parameter value ψ , using the relative efficiency

$$e(\psi) = \left(\frac{E_{\psi}(\text{length of } D^*)}{E_{\theta,\sigma}(\text{length of } J^*)} \right)^2 = \left(\frac{E_{\psi}(\text{length of } D^*)}{2t(n)E(R)} \right)^2.$$

This is a measure of the efficiency of the standard $1 - \alpha$ confidence interval J^* by comparison with the efficiency of the $1 - \alpha$ confidence interval D^* . Farchione and Kabaila [2] present (in Section 6) a computational methodology with a weight function determined by a parameter w , to find a confidence interval D^* such that $e(0)$ is minimized, while ensuring that $\max_{\psi} e(\psi)$ is not too large. In other words, if ψ happens to be 0 then D^* performs better than the standard $1 - \alpha$ confidence interval J^* . On the other hand, if $\psi \neq 0$ then the worst possible performance of

D^* is $\max_{\psi} e(\psi)$, which is not too large. In addition, the confidence interval D^* has endpoints that are the same as the endpoints of the standard $1-\alpha$ confidence interval J^* for sufficiently large $|X|/R$. This implies that $e(\psi) \rightarrow 1$ as $|\psi| \rightarrow \infty$. Farchione and Kabaila [2] found computationally that for the same choice of parameter w , the confidence intervals C^* and D^* have similar relative efficiencies (as function of ψ), provided that n is not small. This is illustrated by Figure 2 of [2]. Theoretical support for this computational finding is provided by Theorem 2 of Appendix B of the present paper.

As in Section 5, suppose that the tuning parameter η_n is chosen so as to lead to conservative model selection. We consider the conditions under which the point estimate

$$\tilde{\theta}_H = \begin{cases} 0 & \text{if } |\bar{y}| \leq \hat{\sigma} \eta_n \\ \bar{y} & \text{if } |\bar{y}| > \hat{\sigma} \eta_n \end{cases}$$

of θ belongs in the confidence interval D (observed value) for all $(\bar{y}, \hat{\sigma}) \in \mathbb{R} \times (0, \infty)$. Define $\tau_n = \sqrt{n} \eta_n$. Multiply the estimate $\tilde{\theta}_H$ and the confidence interval D by $\sqrt{n}/\hat{\sigma}$, to obtain

$$\tilde{\psi}_H = \frac{\sqrt{n}}{\hat{\sigma}} \tilde{\theta}_H = \begin{cases} 0 & \text{if } |\tilde{x}| \leq \tau_n \\ \tilde{x} & \text{if } |\tilde{x}| > \tau_n \end{cases}$$

$$\tilde{D} = \frac{\sqrt{n}}{\hat{\sigma}} D = [-b(-\tilde{x}), b(\tilde{x})],$$

where $\tilde{x} = (\sqrt{n}/\hat{\sigma})\bar{y}$. Obviously, $\tilde{\theta}_H \in D$ for all $(\bar{y}, \hat{\sigma}) \in \mathbb{R} \times (0, \infty)$ is equivalent to $\tilde{\psi}_H \in \tilde{D}$ for all $\tilde{x} \in \mathbb{R}$. There exists a positive number \tilde{c}_H such that, for every $\tau_n \in (0, \tilde{c}_H]$, the following is true: $\tilde{\psi}_H \in \tilde{D}$ for all \tilde{x} . Similar statements hold for the other point estimates $\tilde{\theta}_S$, $\tilde{\theta}_A$ and $\tilde{\theta}_C$. As note earlier, the computational results of [2] and Theorem 2 of Appendix B, suggest that (provided that n is not small) the situation here is very similar to that described in Section 5 and Figures 2 and 3. In other words, we expect that $\tilde{c}_H \approx c_H$ (where c_H is defined in Section 5), provided that n is not small.

7 Conclusion

The results of this paper confirm, yet again, that the hard-thresholding, LASSO, adaptive LASSO and SCAD point estimators form a very poor foundation for confidence interval construction when the tuning parameter for these estimators is chosen to lead to *consistent* model selection. However, the results of this paper do not, by any means, rule out the use of these point estimators as the foundation for confidence interval construction when the tuning parameter for these estimators is chosen to lead to *conservative* model selection.

A Proof of Theorem 1

Define the event $A_n = \{|\bar{Y}_n| \leq \hat{\Sigma}_n \eta_n\}$. By the law of total probability,

$$P_{\theta, \sigma}(\{\theta \in D_n(\bar{Y}_n, \hat{\Sigma}_n)\} \cap A_n) + P_{\theta, \sigma}(\{\theta \in D_n(\bar{Y}_n, \hat{\Sigma}_n)\} \cap A_n^c) \geq 1 - \alpha \text{ for all } (\theta, \sigma).$$

In particular,

$$P_{\theta_n, \sigma}(\{\theta_n \in D_n(\bar{Y}_n, \hat{\Sigma}_n)\} \cap A_n) + P_{\theta_n, \sigma}(\{\theta_n \in D_n(\bar{Y}_n, \hat{\Sigma}_n)\} \cap A_n^c) \geq 1 - \alpha \text{ for all } \sigma.$$

Define the event $B_n = \{u_n(\bar{Y}_n, \hat{\Sigma}_n) \geq \theta_n\}$. When the event A_n occurs, $\ell_n(\bar{Y}_n, \hat{\Sigma}_n) \leq 0$ and so

$$P_{\theta_n, \sigma}(\{\theta_n \in D_n(\bar{Y}_n, \hat{\Sigma}_n)\} \cap A_n) = P_{\theta_n, \sigma}(B_n \cap A_n) \text{ for all } \sigma.$$

Thus, for each $\sigma \in (0, \infty)$,

$$\begin{aligned} P_{\theta_n, \sigma}(B_n \cap A_n) &\geq 1 - \alpha - P_{\theta_n, \sigma}(\{\theta_n \in D_n(\bar{Y}_n, \hat{\Sigma}_n)\} \cap A_n^c) \\ &\geq 1 - \alpha - P_{\theta_n, \sigma}(A_n^c). \end{aligned}$$

Lemma 1. For each $\sigma \in (0, \infty)$, $P_{\theta_n, \sigma}(A_n^c) \rightarrow 0$ as $n \rightarrow \infty$.

Proof. Fix $\sigma \in (0, \infty)$. It is sufficient to prove that $P_{\theta_n, \sigma}(A_n) \rightarrow 1$ as $n \rightarrow \infty$. Now

$$A_n = \left\{ |X_n| \leq \frac{\hat{\Sigma}_n}{\sigma} \sqrt{n} \eta_n \right\}$$

where $X_n = \sqrt{n} \bar{Y}_n / \sigma$. Note that $X_n \sim N(\sqrt{n} \eta_n / 2, 1)$. Observe that

$$\left\{ \frac{\hat{\Sigma}_n}{\sigma} > \frac{3}{4} \right\} \cap \left\{ \left| X_n - \frac{1}{2} \sqrt{n} \eta_n \right| \leq \frac{1}{4} \sqrt{n} \eta_n \right\} \subset A_n.$$

Thus

$$\begin{aligned} P_{\theta_n, \sigma}(A_n) &\geq P_{\theta_n, \sigma} \left(\left\{ \frac{\hat{\Sigma}_n}{\sigma} > \frac{3}{4} \right\} \cap \left\{ \left| X_n - \frac{1}{2} \sqrt{n} \eta_n \right| \leq \frac{1}{4} \sqrt{n} \eta_n \right\} \right) \\ &= P_{\theta_n, \sigma} \left(\frac{\hat{\Sigma}_n}{\sigma} > \frac{3}{4} \right) P_{\theta_n, \sigma} \left(\left| X_n - \frac{1}{2} \sqrt{n} \eta_n \right| \leq \frac{1}{4} \sqrt{n} \eta_n \right) \end{aligned}$$

and the right-hand-side converges to 1 as $n \rightarrow \infty$. □

Also, when the event $B_n \cap A_n$ occurs, $\ell_n(\bar{Y}_n, \hat{\Sigma}_n) \leq 0$ and $u_n(\bar{Y}_n, \hat{\Sigma}_n) \geq \theta_n$, so that $u_n(\bar{Y}_n, \hat{\Sigma}_n) - \ell_n(\bar{Y}_n, \hat{\Sigma}_n) \geq \theta_n$. Hence,

$$E_{\theta_n, \sigma}(\text{length of } D_n(\bar{Y}_n, \hat{\Sigma}_n)) \geq P_{\theta_n, \sigma}(B_n \cap A_n) \theta_n.$$

Thus, for each $\sigma \in (0, \infty)$,

$$\begin{aligned} \frac{E_{\theta_n, \sigma}(\text{length of } D_n(\bar{Y}_n, \hat{\Sigma}_n))}{E_{\theta_n, \sigma}(\text{length of standard } 1 - \alpha \text{ CI for } \theta)} &\geq \frac{P_{\theta_n, \sigma}(B_n \cap A_n) \theta_n}{2 t(n-1) E(\hat{\Sigma}_n) / \sqrt{n}} \\ &= \frac{P_{\theta_n, \sigma}(B_n \cap A_n) \sqrt{n} \eta_n}{4 t(n-1) E(\hat{\Sigma}_n / \sigma)}, \end{aligned}$$

which tends to infinity as $n \rightarrow \infty$.

B The motivation for considering the known error variance case

In this appendix, we motivate the consideration of the “known error variance case”.

We begin by supposing that the error variance σ^2 is unknown and is estimated by $\hat{\sigma}^2$. We apply the same change of scale (by multiplying by \sqrt{n}/σ) to the parameter θ , the estimator \bar{Y} and the estimators $\tilde{\Theta}_H$, $\tilde{\Theta}_S$, $\tilde{\Theta}_A$ and $\tilde{\Theta}_C$ as follows.

Define $\psi = (\sqrt{n}/\sigma) \theta$, $X = (\sqrt{n}/\sigma) \bar{Y}$, $\tau_n = \sqrt{n}\eta_n$ and $R = \hat{\Sigma}/\sigma$. Note that X and R are independent random variables and that $X \sim N(\psi, 1)$. Also define

$$\tilde{\Psi}_H = \frac{\sqrt{n}}{\sigma} \tilde{\Theta}_H = \begin{cases} 0 & \text{if } |X| \leq R\tau_n \\ X & \text{if } |X| > R\tau_n \end{cases}$$

$$\tilde{\Psi}_S = \frac{\sqrt{n}}{\sigma} \tilde{\Theta}_S = \begin{cases} -\max\{|X| - R\tau_n, 0\} & \text{if } X < 0 \\ 0 & \text{if } X = 0 \\ \max\{|X| - R\tau_n, 0\} & \text{if } X > 0 \end{cases}$$

$$\tilde{\Psi}_A = \frac{\sqrt{n}}{\sigma} \tilde{\Theta}_A = \begin{cases} 0 & \text{if } |X| \leq R\tau_n \\ X - \frac{R^2\tau_n^2}{X} & \text{if } |X| > R\tau_n \end{cases}$$

$$\tilde{\Psi}_C = \frac{\sqrt{n}}{\sigma} \tilde{\Theta}_C = \begin{cases} \text{sign}(X)(|X| - R\tau_n)_+ & \text{if } |X| \leq 2R\tau_n \\ ((a-1)X - \text{sign}(X)aR\tau_n)/(a-2) & \text{if } 2R\tau_n < |X| \leq aR\tau_n \\ X & \text{if } |X| > aR\tau_n \end{cases}$$

These are not estimators of ψ since they depend on the unknown parameter σ . Since R and X are independent and R converges in probability to 1 (as $n \rightarrow \infty$) it is plausible that, for large n , the statistical properties of $\tilde{\Psi}_H$, $\tilde{\Psi}_S$, $\tilde{\Psi}_A$ and $\tilde{\Psi}_C$ are well-approximated by these properties of the corresponding quantities:

$$\hat{\Psi}_H = \begin{cases} 0 & \text{if } |X| \leq \tau_n \\ X & \text{if } |X| > \tau_n \end{cases}$$

$$\hat{\Psi}_S = \begin{cases} -\max\{|X| - \tau_n, 0\} & \text{if } X < 0 \\ 0 & \text{if } X = 0 \\ \max\{|X| - \tau_n, 0\} & \text{if } X > 0 \end{cases}$$

$$\hat{\Psi}_A = \begin{cases} 0 & \text{if } |X| \leq \tau_n \\ X - \frac{\tau_n^2}{X} & \text{if } |X| > \tau_n \end{cases}$$

$$\hat{\Psi}_C = \begin{cases} \text{sign}(X)(|X| - \tau_n)_+ & \text{if } |X| \leq 2\tau_n \\ ((a-1)X - \text{sign}(X)a\tau_n)/(a-2) & \text{if } 2\tau_n < |X| \leq a\tau_n \\ X & \text{if } |X| > a\tau_n \end{cases}$$

Note that, conveniently, the statistical properties of these quantities depend only on the parameter ψ and not on the parameter σ .

Farchione and Kabaila [2] consider the following confidence interval for θ :

$$D = \left[-\frac{\hat{\Sigma}}{\sqrt{n}} b\left(-\frac{\bar{Y}}{\hat{\Sigma}/\sqrt{n}}\right), \frac{\hat{\Sigma}}{\sqrt{n}} b\left(\frac{\bar{Y}}{\hat{\Sigma}/\sqrt{n}}\right) \right]$$

where the function b must satisfy the constraint that $b(x) \geq -b(-x)$ for all $x \in \mathbb{R}$. This constraint is required to ensure that the upper endpoint of this confidence interval is never less than the lower endpoint. This particular form of confidence interval is motivated by some invariance arguments. The standard $1 - \alpha$ confidence interval for θ is

$$[\bar{Y} - t(n-1)\hat{\Sigma}/\sqrt{n}, \bar{Y} + t(n-1)\hat{\Sigma}/\sqrt{n}]$$

where the quantile $t(m)$ is defined by the requirement that $P(-t(m) \leq T \leq t(m)) = 1 - \alpha$ for $T \sim t_m$. Note that this confidence interval can be expressed in the form D .

Now scale the confidence interval D by the same scaling factor as before, to obtain

$$D^* = \frac{\sqrt{n}}{\sigma} D = [-R b(-X/R), R b(X/R)].$$

Note that $\tilde{\Theta}_H \in D$ is equivalent to $\tilde{\Psi}_H \in D^*$. Similar statements apply to the other estimators $\tilde{\Theta}_S$, $\tilde{\Theta}_A$ and $\tilde{\Theta}_C$. Also note that D^* is not a confidence interval for ψ , since it depends on the unknown parameter σ . However,

$$P_{\theta,\sigma}(\theta \in D) = P_{\psi}(\psi \in D^*),$$

so that

$$\inf_{\theta,\sigma} P_{\theta,\sigma}(\theta \in D) = \inf_{\psi} P_{\psi}(\psi \in D^*).$$

Also,

$$\frac{E_{\theta,\sigma}(\text{length of } D)}{E_{\theta,\sigma}(\text{length of standard } 1 - \alpha \text{ CI for } \theta)} = \frac{E_{\psi}(\text{length of } D^*)}{2t(n-1)E(R)}.$$

Since R and X are independent and R converges in probability to 1 (as $n \rightarrow \infty$) it is plausible that, for large n , the statistical properties of D^* are well-approximated by the corresponding properties of $C^* = [-b(-X), b(X)]$. In fact, the following result holds.

Theorem 2. *Suppose that the function b satisfies the following assumptions.*

(A1) *The function b is continuous and strictly increasing. Also, the function b^{-1} is uniformly continuous.*

(A2) *Define $e(x) = b(x) - x - z$, where the quantile z is defined by the requirement that $P(-z \leq Z \leq z) = 1 - \alpha$ for $Z \sim N(0, 1)$.*

(i) *$e(x) = 0$ for all $|x| \geq q$, where q is a specified positive number.*

(ii) *There exists L , satisfying $0 < L < \infty$, such that $|e(x) - e(y)| \leq L|x - y|$ for all x and y .*

Then

(R1) $\sup_{\psi} |P_{\psi}(\psi \in C^*) - P_{\psi}(\psi \in D^*)| \rightarrow 0$ as $n \rightarrow \infty$.

(R2) $\sup_{\psi} \left| \frac{E_{\psi}(\text{length of } C^*)}{2z} - \frac{E_{\psi}(\text{length of } D^*)}{2t(n-1)E(R)} \right| \rightarrow 0$ as $n \rightarrow \infty$.

Proof. We prove the result (R1) as follows. Note that

$$P_{\psi}(\psi \in C^*) = 1 - P_{\psi}(\psi < -b(-X)) - P_{\psi}(\psi > b(X))$$

$$P_{\psi}(\psi \in D^*) = 1 - P_{\psi}(\psi < -Rb(-X/R)) - P_{\psi}(\psi > Rb(X/R)).$$

It is sufficient to prove that

$$\sup_{\psi} |P_{\psi}(\psi < -b(-X)) - P_{\psi}(\psi < -Rb(-X/R))| \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad (3)$$

$$\sup_{\psi} |P_{\psi}(\psi > b(X)) - P_{\psi}(\psi > Rb(X/R))| \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad (4)$$

The proofs of (3) and (4) are very similar. For the sake of brevity, we provide only the proof of (4). Suppose that $\epsilon > 0$ is given. We need to prove that there exists $N < \infty$ such that

$$\sup_{\psi} |P_{\psi}(\psi > b(X)) - P_{\psi}(\psi > Rb(X/R))| < \epsilon \quad \text{for all } n > N. \quad (5)$$

Let δ ($0 < \delta < 1/2$) be given. Using the law of total probability, it may be shown that

$$\begin{aligned} & |P_{\psi}(\psi > Rb(X/R)) - P_{\psi}(\psi > b(X))| \\ & \leq |P_{\psi}(\psi > Rb(X/R), |R - 1| \leq \delta) - P_{\psi}(\psi > b(X))| + P(|R - 1| > \delta). \end{aligned} \quad (6)$$

Obviously,

$$\begin{aligned} & P_{\psi}(\psi > Rb(X/R), |R - 1| \leq \delta) \\ & = P_{\psi}(\psi > b(X) + (R - 1)(z + e(X/R)) + (e(X/R) - e(X)), |R - 1| \leq \delta). \end{aligned} \quad (7)$$

It may be shown that if $|R - 1| \leq \delta$ then there exists $M < \infty$ (where M does not depend on δ) such that $|(R - 1)(z + e(X/R)) + (e(X/R) - e(X))| \leq M\delta$. Thus

$$P_{\psi}(\psi > b(X) + M\delta, |R - 1| \leq \delta) \leq (7) \leq P_{\psi}(\psi > b(X) - M\delta).$$

Using the law of total probability, it may be shown that

$$P_{\psi}(\psi > b(X) + M\delta, |R - 1| \leq \delta) \leq (7) \geq P_{\psi}(\psi > b(X) + M\delta) - P(|R - 1| > \delta).$$

Thus

$$P_{\psi}(\psi > b(X) + M\delta) - P(|R - 1| > \delta) \leq (7) \leq P_{\psi}(\psi > b(X) - M\delta).$$

In other words,

$$P_{\psi}(X < b^{-1}(\psi - M\delta)) - P(|R - 1| > \delta) \leq (7) \leq P_{\psi}(X < b^{-1}(\psi + M\delta)).$$

Note that $P_\psi(\psi > b(X)) = P_\psi(X < b^{-1}(\psi))$. Using the uniform continuity of b^{-1} and the fact that $X \sim N(\psi, 1)$, it may be shown that there exists δ ($0 < \delta < 1/2$) such that

$$\begin{aligned} \sup_{\psi} |P_\psi(X < b^{-1}(\psi - M\delta)) - P_\psi(X < b^{-1}(\psi))| &< \epsilon/2 \\ \sup_{\psi} |P_\psi(X < b^{-1}(\psi + M\delta)) - P_\psi(X < b^{-1}(\psi))| &< \epsilon/2. \end{aligned}$$

Choose δ ($0 < \delta < 1/2$) such that these two inequalities are satisfied. Therefore, $|(7) - P_\psi(\psi > b(X))| < P(|R - 1| > \delta) + \epsilon/2$. It follows from (6) that $|P_\psi(\psi > Rb(X/R)) - P_\psi(\psi > b(X))| < 2P(|R - 1| > \delta) + \epsilon/2$. Since $P(|R - 1| > \delta) \rightarrow 0$ as $n \rightarrow \infty$, there exists $N < \infty$ such that (5) is satisfied. This completes the proof of the result (R1).

We prove the result (R2) as follows. It may be shown that it is sufficient to prove that

$$\sup_{\psi} |E_\psi(\text{length of } C^*) - E_\psi(\text{length of } D^*)| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Now

$$\begin{aligned} &E_\psi(\text{length of } D^*) - E_\psi(\text{length of } C^*) \\ &= 2z(E(R) - 1) + (E_\psi(\text{length of } D^*) - 2zE(R)) - (E_\psi(\text{length of } C^*) - 2z). \end{aligned}$$

Hence

$$\begin{aligned} &|E_\psi(\text{length of } D^*) - E_\psi(\text{length of } C^*)| \\ &= 2z|E(R) - 1| + |E_\psi(\text{length of } D^*) - 2zE(R) - (E_\psi(\text{length of } C^*) - 2z)|. \end{aligned}$$

Since $E(R)$ does not depend on ψ and $E(R) \rightarrow 1$ as $n \rightarrow \infty$, it is sufficient to prove that

$$\sup_{\psi} |E_\psi(\text{length of } D^*) - 2zE(R) - (E_\psi(\text{length of } C^*) - 2z)| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Let f_R denote the probability density function of R . Now

$$\begin{aligned}
& E_\psi(\text{length of } D^*) - 2zE(R) \\
&= \int_0^\infty \int_{-\infty}^\infty \left(b\left(\frac{x}{r}\right) + b\left(-\frac{x}{r}\right) - 2z \right) \phi(x - \psi) dx r f_R(r) dr \\
&= \int_0^\infty \int_{-rq}^{rq} \left(b\left(\frac{x}{r}\right) + b\left(-\frac{x}{r}\right) - 2z \right) \phi(x - \psi) dx r f_R(r) dr \tag{8}
\end{aligned}$$

since $b(x/r) + b(-x/r) - 2z = 0$ for all $|x| \geq rq$. Changing the variable of integration from x to $y = x/r$, we see that (8) is equal to

$$\int_0^\infty \int_{-q}^q (b(x) + b(-x) - 2z) \phi(rx - \psi) dx r^2 f_R(r) dr$$

Now

$$\begin{aligned}
& E_\psi(\text{length of } C^*) - 2z \\
&= \int_{-\infty}^\infty (b(x) + b(-x) - 2z) \phi(x - \psi) dx \\
&= \int_{-q}^q (b(x) + b(-x) - 2z) \phi(x - \psi) dx \quad (\text{by (A2)(i)}) \\
&= \int_0^\infty \int_{-q}^q (b(x) + b(-x) - 2z) \phi(x - \psi) dx r^2 f_R(r) dr, \tag{9}
\end{aligned}$$

since

$$\int_0^\infty r^2 f_R(r) dr = E(R^2) = 1.$$

Thus

$$\begin{aligned}
& E_\psi(\text{length of } D^*) - 2zE(R) - (E_\psi(\text{length of } C^*) - 2z) \\
&= \int_0^\infty \int_{-q}^q (e(x) + e(-x)) (\phi(rx - \psi) - \phi(x - \psi)) dx r^2 f_R(r) dr \tag{10}
\end{aligned}$$

By the mean-value theorem, there exists a positive number $K < \infty$ such that

$$|\phi(rx - \psi) - \phi(x - \psi)| \leq K|r - 1||x| \quad \text{for all } r \geq 0 \text{ and } x \in \mathbb{R}.$$

Thus

$$|(10)| \leq 4LKq^3 E(|R - 1|R^2).$$

Note that $E(|R - 1|R^2)$ does not depend on ψ and that, by the Cauchy-Schwarz inequality, $E(|R - 1|R^2) \rightarrow 0$ as $n \rightarrow \infty$. This completes the proof of (R2).

□

Thus, to study the coverage and expected length properties of the confidence interval D for θ when n is large, we study the properties of $P_\psi(\psi \in C^*)$ and $E_\psi(\text{length of } C^*)$, which are simply functions of ψ .

Now suppose that the “error variance is known” i.e. σ^2 is known. The analogues of the estimators $\tilde{\Theta}_H$, $\tilde{\Theta}_S$, $\tilde{\Theta}_A$ and $\tilde{\Theta}_C$ are $\hat{\Theta}_H$, $\hat{\Theta}_S$, $\hat{\Theta}_A$ and $\hat{\Theta}_C$ respectively, where

$$\begin{aligned}\hat{\Theta}_H &= \begin{cases} 0 & \text{if } |\bar{Y}| \leq \sigma \eta_n \\ \bar{Y} & \text{if } |\bar{Y}| > \sigma \eta_n \end{cases} \\ \hat{\Theta}_S &= \begin{cases} -\max\{|\bar{Y}| - \sigma \eta_n, 0\} & \text{if } \bar{Y} < 0 \\ 0 & \text{if } \bar{Y} = 0 \\ \max\{|\bar{Y}| - \sigma \eta_n, 0\} & \text{if } \bar{Y} > 0 \end{cases} \\ \hat{\Theta}_A &= \begin{cases} 0 & \text{if } |\bar{Y}| \leq \sigma \eta_n \\ \bar{Y} - \frac{\sigma^2 \eta_n^2}{\bar{Y}} & \text{if } |\bar{Y}| > \sigma \eta_n \end{cases} \\ \hat{\Theta}_C &= \begin{cases} \text{sign}(\bar{Y}) (|\bar{Y}| - \sigma \eta_n)_+ & \text{if } |\bar{Y}| \leq 2\sigma \eta_n \\ ((a-1)\bar{Y} - \text{sign}(\bar{Y})a\sigma \eta_n)/(a-2) & \text{if } 2\sigma \eta_n < |\bar{Y}| \leq a\sigma \eta_n \\ \bar{Y} & \text{if } |\bar{Y}| > a\sigma \eta_n \end{cases}\end{aligned}$$

where $a = 3.7$. Also, the analogue of the confidence interval D for θ is

$$C = \left[-\frac{\sigma}{\sqrt{n}} b \left(-\frac{\bar{Y}}{\sigma/\sqrt{n}} \right), \frac{\sigma}{\sqrt{n}} b \left(\frac{\bar{Y}}{\sigma/\sqrt{n}} \right) \right].$$

Scaling θ , $\hat{\Theta}_H$, $\hat{\Theta}_S$, $\hat{\Theta}_A$, $\hat{\Theta}_C$ and C by multiplying by \sqrt{n}/σ , we obtain ψ , $\hat{\Psi}_H$, $\hat{\Psi}_S$, $\hat{\Psi}_A$, $\hat{\Psi}_C$ and C^* , respectively. In other words, when we suppose that the “error variance is known”, we are finding an approximation (by the arguments stated earlier in this section) to the coverage probability and expected length properties of $\tilde{\Theta}_H$, $\tilde{\Theta}_S$, $\tilde{\Theta}_A$, $\tilde{\Theta}_C$ and D for large n .

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